An Efficient Continuous in Time Data Assimilation Algorithm for the Sabra Shell Model of Turbulence

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Introduction

- Complex anisotropic turbulent systems
  - ubiquitous in geoscience, engineering and climate science
  - strong intermittent instabilities
  - partial observations
- An efficient continuous data assimilation algorithm is developed for estimating the unobserved state and the associated uncertainty.
- The new data assimilation scheme is combined with a simple reduced order modeling technique.
- The new data assimilation scheme is then applied to the Sabra shell model.

Nonlinear Conditional Gaussian Systems

Despite the fully nonlinearity in many multiscale turbulent dynamical systems and the non-Gaussian features in both the marginal and joint PDFs, these systems have conditional Gaussian structures.

The general nonlinear conditional Gaussian systems

$$\frac{\mathrm{d}\mathbf{v}}{\mathrm{d}t} = \mathbf{A_0}(\mathbf{v},t) + \mathbf{A_1}(\mathbf{v},t)\mathbf{w} + \boldsymbol{\sigma}_v(\mathbf{v},t)\dot{\mathbf{W}}_v, \tag{1a}$$

$$\frac{\mathrm{d}\mathbf{w}}{\mathrm{d}t} = \mathbf{a_0}(\mathbf{v},t) + \mathbf{a_1}(\mathbf{v},t)\mathbf{w} + \boldsymbol{\sigma}_w(\mathbf{v},t)\dot{\mathbf{W}}_w, \tag{1b}$$

conditional on one realization (i.e., a random trajectory) of $\mathbf{v}$, the conditional distribution

$$p\big(\mathbf{w}(t)|\mathbf{v}(s \le t)\big) \sim \mathcal{N}(\boldsymbol{\mu}, \mathbf{R}) \tag{2}$$

is Gaussian.

- Despite the conditional Gaussianity, the coupled system 1 remains highly nonlinear and is able to capture the non-Gaussian features as in nature.
- The conditional Gaussian distribution in 2 has closed analytic form:

$$\mathrm{d}\boldsymbol{\mu} = (\mathbf{a_0} + \mathbf{a_1}\boldsymbol{\mu})\,\mathrm{d}t + \mathbf{R}\mathbf{A}_\mathbf{1}^*(\boldsymbol{\sigma}_v\boldsymbol{\sigma}_v^*)^{-1}(\,\mathrm{d}\mathbf{v} - (\mathbf{A_0} + \mathbf{A_1}\boldsymbol{\mu})\,\mathrm{d}t), \tag{3a}$$

$$\mathrm{d}\mathbf{R} = \big(\mathbf{a_1}\mathbf{R} + \mathbf{R}\mathbf{a}_\mathbf{1}^* + \boldsymbol{\sigma}_w\boldsymbol{\sigma}_w^* - \mathbf{R}\mathbf{A}_\mathbf{1}^*(\boldsymbol{\sigma}_v\boldsymbol{\sigma}_v^*)^{-1}\mathbf{A_1}\mathbf{R}\big)\,\mathrm{d}t, \tag{3b}$$

Incorporating PDE into the Data Assimilation Framework

Given a set of basis functions $\{\boldsymbol{\varphi}_1, \boldsymbol{\varphi}_2, \ldots, \boldsymbol{\varphi}_R\}$. Projecting the PDE $\mathcal{M}$ onto these basis leads to

$$\frac{\mathrm{d}\mathbf{u}_R}{\mathrm{d}t} = \mathbf{A}_u \mathbf{u}_R + \mathbf{u}_R^* \mathbf{B}_u \mathbf{u}_R + \text{residual}, \tag{4}$$

where $\mathbf{u}_R = (\hat{u}_1, \ldots, \hat{u}_R)^\mathtt{T}$ is the collection of the state variables in the projection space with $\mathbf{u}_R = \sum_{i=1}^{R} \hat{u}_i \boldsymbol{\varphi}_i$ being the approximate solution of $\mathcal{M}$.
Next, the R modes of $\mathbf{u}_R$ are categorized into two groups:

- $\mathbf{v}$ the observed variables
- $\mathbf{w}$ the unobserved variables

Therefore, the system (4) can be rewritten as

$$\frac{d\mathbf{v}}{dt} = \mathbf{A}_v^{(v)} \mathbf{v} + \mathbf{v}^* \mathbf{B}_{vv}^{(v)} \mathbf{v} + \mathbf{v}^* \mathbf{B}_{vw}^{(v)} \mathbf{w} + \mathbf{w}^* \mathbf{B}_{ww}^{(v)} \mathbf{w} + \text{residual}_1, \tag{5a}$$

$$\frac{d\mathbf{w}}{dt} = \mathbf{A}_w^{(w)} \mathbf{w} + \mathbf{v}^* \mathbf{B}_{vv}^{(w)} \mathbf{v} + \mathbf{v}^* \mathbf{B}_{vw}^{(w)} \mathbf{w} + \mathbf{w}^* \mathbf{B}_{ww}^{(w)} \mathbf{w} + \text{residual}_2, \tag{5b}$$

where the $\mathbf{B}_{\cdot}^{(\cdot)}$ in (5) are related to the $\mathbf{B}_u$ in (4).

Incorporating PDE into the Data Assimilation Framework

Finally, to get the conditional Gaussian structure, the self-interaction of $\mathbf{w}$, namely the quadratic nonlinearity between the unobserved variables themselves, together with the residual terms are dropped. To compensate, additional parameterizations and stochastic noise are utilized to approximate the contribution from these terms, namely,

$$\mathbf{w}^*\mathbf{B}_{ww}^{(v)}\mathbf{w} + \text{residual}_1 \approx \boldsymbol{\tau}_v^{(v)}(\mathbf{v}) + \boldsymbol{\sigma}_v\dot{\mathbf{W}}_v, \tag{6a}$$

$$\mathbf{w}^*\mathbf{B}_{ww}^{(w)}\mathbf{w} + \text{residual}_2 \approx \boldsymbol{\tau}_v^{(w)}(\mathbf{v}) + \boldsymbol{\sigma}_w\dot{\mathbf{W}}_w. \tag{6b}$$

- Motivation
  - These terms represent the self-interactions between high frequencies. Stochastic noise is a suitable surrogate to approximate the fast variabilities.
  - The unobserved variables in general may not contain only the fast components. It's essential to further include additional deterministic parameterizations.
  - Desired mathematical structure (conditional Gaussian).

Determining the Parameterization and Noise Coefficient

- Multivariate polynomial regression (MPR) method is used to determine the terms $\boldsymbol{\tau}_v^{(v)}(\mathbf{v})$ and $\boldsymbol{\tau}_v^{(w)}(\mathbf{v})$, where both terms are assumed to be a quadratic polynomial of $\mathbf{v}$.
- Noise coefficients $\boldsymbol{\sigma}_v$ and $\boldsymbol{\sigma}_w$ are determined by the standard deviation of the residual between the truth and the MPR fit.

Incorporating PDE into the Data Assimilation Framework

Collecting all the above information yields the following coupled model,

$$\frac{d\mathbf{v}}{dt} = \mathbf{A}_v^{(v)}\mathbf{v} + \mathbf{v}^*\mathbf{B}_{vv}^{(v)}\mathbf{v} + \mathbf{v}^*\mathbf{B}_{vw}^{(v)}\mathbf{w} + \boldsymbol{\tau}_v^{(v)}(\mathbf{v}) + \boldsymbol{\sigma}_v\dot{\mathbf{W}}_v, \tag{7a}$$

$$\frac{d\mathbf{w}}{dt} = \mathbf{A}_w^{(w)}\mathbf{w} + \mathbf{v}^*\mathbf{B}_{vv}^{(w)}\mathbf{v} + \mathbf{v}^*\mathbf{B}_{vw}^{(w)}\mathbf{w} + \boldsymbol{\tau}_v^{(w)}(\mathbf{v}) + \boldsymbol{\sigma}_w\dot{\mathbf{W}}_w. \tag{7b}$$

Model

- Sabra shell model

$$\frac{du_n}{dt} + \nu k_n^2 u_n = i(ak_{n+1}u_{n+1}^* u_{n+2} + bk_n u_{n-1}^* u_{n+1} - ck_{n-1}u_{n-1}u_{n-2}) + f_n, \tag{8}$$

$n = 1 \ldots N$, $a + b + c = 0$, $k_n = k_0 \lambda^n$.

  - Structurally similar spectral properties as that of 3D Navier-Stokes equations.
  - Highly reduced degrees of freedom.
  - Ability to numerically reproduce intermittencies.

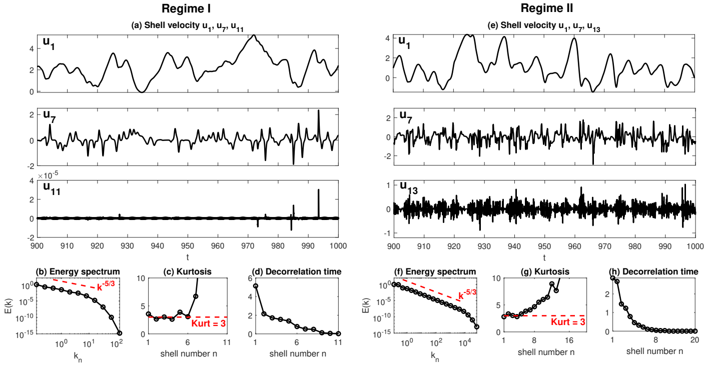
- Model parameters
  - Intershell ratio given by $\lambda = 2$, so that $k_n = k_0 \lambda^n$, with $k_0 = 2^{-4}$. The interaction coefficients are set to $a = 1, b = c = -1/2$. Constant forcing with magnitude one are imposed onto the first two shells $n = 1$ and $n = 2$.

- Observed and unobserved variables
  -

$$\begin{aligned} \mathbf{v} &= (u_1, u_2, u_5, u_6) \\ \mathbf{w} &= (u_3, u_4, u_7, u_8). \end{aligned} \tag{9}$$

Dynamical regimes

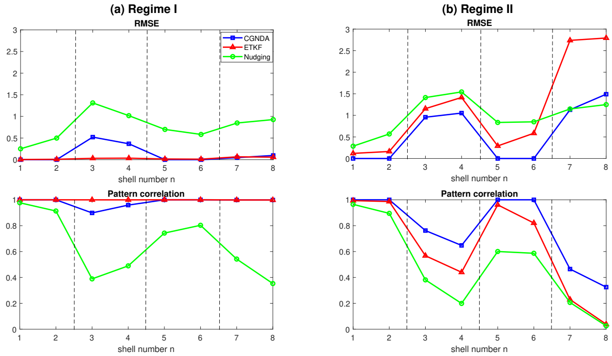
Two dynamical regimes, which are differed by the viscosity coefficient ν, are studied here. Regime I corresponds to a moderate viscosity $\nu = 0.09$ with a total number of the shells being $N = 11$. Regime II involves a tiny viscosity $\nu = 10^{-5}$ and the total number of the shells is $N = 20$.



- Non-Gaussian features, including the intermittency and extreme events, are clearly illustrated in the time series

Data Assimilation results

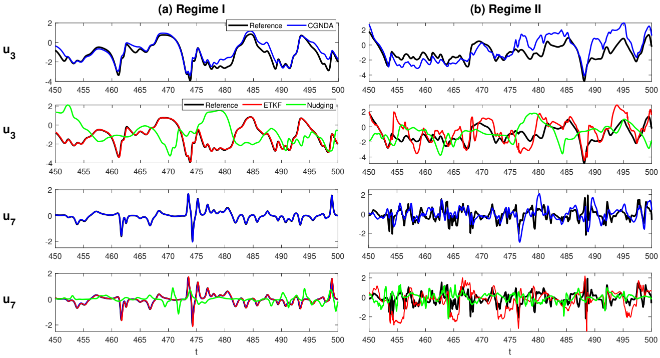
We compare the data assimilation skill of conditional Gaussian nonlinear data assimilation (CGNDA) with the ensemble Kalman filter (EnKF) and nudging (also known as Newtonian Relaxation).



- ETKF is at least 20 times more expensive than CGNDA in both regimes.
- Only an 8-dimensional reduced order system is utilized in the CGNDA while the full perfect system is adopted for the ETKF.

Data Assimilation results

- Comparison of the posterior mean time series with the reference solution (only the real part is shown here).

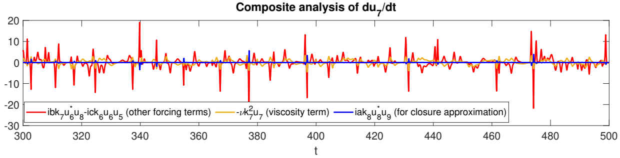


Data Assimilation results

Why the recovered small-scale variables u_7 and u_8 using CGNDA are almost identical to the reference solution?

$$\frac{du_7}{dt} = \underbrace{-\nu k_7^2 u_7}_{\{1\}} + \underbrace{iak_8 u_8^* u_9}_{\{2\}} + \underbrace{ibk_7 u_6^* u_8 - ick_6 u_6 u_5}_{\{3\}}. \tag{10}$$

$\{2\}$ which involves higher order shells is approximated by the closure term $\boldsymbol{\tau}_v^{(w)}(\mathbf{v})$. The term $\{1\}$ is the viscous term, which damps the signal. The term $\{3\}$ does not involve u_7 itself, and therefore it can be regarded as an external forcing term.



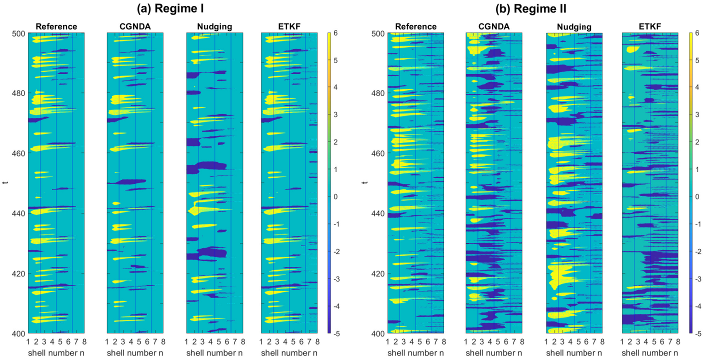
- The energy associated with $\{1\}$, $\{2\}$ and $\{3\}$ is 11.5647%, 0.63% and 87.8052%.
- This means the approximate error is damped immediately.

Recovery of the energy flux

The nonlinear flux through a shell n denoted by Π_n can be computed as the difference of nonlinear transfers involving only two triads,

$$\Pi_n = k_n \Im(u_n^* u_{n+1}^* u_{n+2}) - (\epsilon - 1) k_{n-1} \Im(u_{n-1}^* u_n^* u_{n+1}) \tag{11}$$

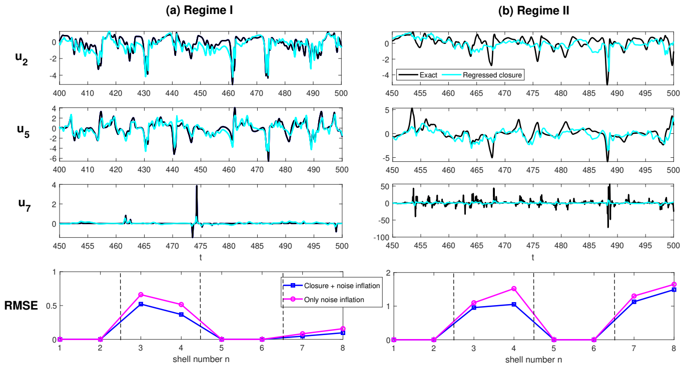
where $\Im$ denotes the imaginary part of the expression and $\epsilon = 1/2$ is used here.



- The energy flux in shell numbers 2 to 4 obtained from the CGDNA closely resembles that of the reference solution in several different time intervals.
- The intermittent bursts of energy transfer in wavenumbers 6 to 8 for CGDNA also have a nearly perfect match with the arrival of these bursts in the reference.

Closure approximation and noise inflation

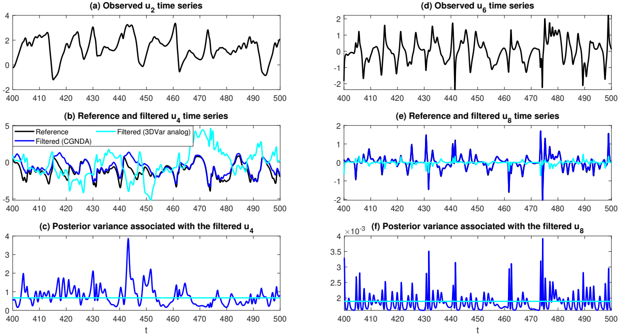
The accuracy in applying such a closure approximation and the necessity of the closure terms.



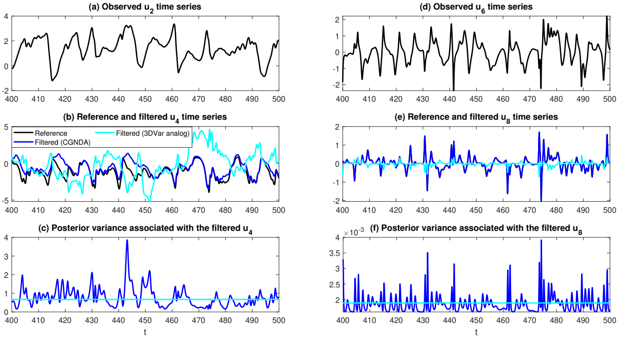
- For the observed variables u_2 and u_5, the closure illustrates a high skill in approximating the truth.
- The intermittent events are captured quite accurately in both the regimes.
- Comparing with the CGNDA with the closure approximation, it is obvious that using only the noise inflation is not as skillful as the closure approximation, which indicates the necessity of the closure terms.

Suboptimal data assimilation

The CGNDA has one advantage that the posterior covariance is a time-dependent function. In comparison, also as an analog to the 3DVar algorithm, we set the posterior covariance to be a constant.



Suboptimal data assimilation



- The peaks of the posterior covariance align well with the extreme events in the unobserved time series (especially for u_8).
- Indicates the necessity in recovering the intermittent features using such a non-stationary uncertainty evolution.

Conclusion

- An efficient continuous data assimilation scheme, the CGNDA, is developed.
- A simple approximation modeling framework is utilized to allow a starting PDE system to satisfy the mathematical structure of the CGNDA scheme.
- The new algorithm is applied to the Sabra shell model, which is a conceptual model for turbulence. It has been shown that the CGNDA outweighs both the ETKF and the nudging data assimilation schemes in terms of both the accuracy and the computational efficiency.